

RUIN PROBABILITIES IN TOUGH TIMES

Part 2

HEAVY-TRAFFIC APPROXIMATION FOR FRACTIONALLY DIFFERENTIATED RANDOM WALKS IN THE DOMAIN OF ATTRACTION OF A NONGAUSSIAN STABLE DISTRIBUTION

Ph. Barbe⁽¹⁾ and W.P. McCormick⁽²⁾
(⁽¹⁾CNRS (UMR 8088), (⁽²⁾University of Georgia

Abstract. Motivated by applications to insurance mathematics, we prove some heavy-traffic limit theorems for processes which encompass the fractionally differentiated random walk as well as some FARIMA processes, when the innovations are in the domain of attraction of a nonGaussian stable distribution.

AMS 2010 Subject Classifications: Primary: 60F99; Secondary: 60G52, 60G22, 60K25, 62M10, 60G70, 62P20.

Keywords: heavy traffic, ruin probability, fractional random walk, FARIMA process, Poisson process.

1. Introduction and main result. The purpose of this paper is to complete our study of ruin probability for some nonstationary processes having long range dependence and innovations in the domain of attraction of a nonGaussian stable distribution, when the premium can hardly cover the claims. The overall motivations for this study are described in the first part of this work (Barbe and McCormick, 2010b).

To recall the setting, we consider a function g analytic on $(-1, 1)$,

$$g(x) = \sum_{i \geq 0} g_i x^i.$$

Given a distribution function F , we can define a (g, F) -process (S_n) as follows. Let $(X_i)_{i \geq 1}$ be a sequence of random variables, independent, having common distribution function F . We set $X_i = 0$ if i is nonpositive. We define the backward shift operator B by $BX_i = X_{i-1}$ and set

$$S_n = g(B)X_n = \sum_{0 \leq i < n} g_i X_{n-i}.$$

Important examples of such processes include

- random walks: $g(x) = (1 - x)^{-1}$,
- ARMA processes: g a rational function with poles outside the complex unit disk, and
- FARIMA processes: $g(x) = (1 - x)^{-\gamma}R(x)$ where γ is positive and R is a rational function not vanishing at 1 and with poles outside the complex unit disk.

In particular, if $g(x) = (1 - x)^{-\gamma}$, then

$$S_n = (1 - B)^{1-\gamma}(1 - B)^{-1}X_n$$

is the random walk $(1 - B)^{-1}X_n$ differentiated $1 - \gamma$ times.

Following the first part of this work, we assume that the sequence

$$(g_n) \text{ is ultimately positive and regularly varying of } \\ \text{negative index } \gamma - 1 \text{ with } \gamma \text{ in } (0, 1). \quad (1.1)$$

This forces the function g to diverge to $+\infty$ as its argument tends to 1. Part 1 of this work concentrates on the case where γ is greater than 1, forcing (g_n) to diverge to infinity. In contrast, in this current paper, we assume that γ is less than 1, forcing (g_n) to converge to 0.

For any positive t define the partial sum

$$g_{[0,t)} = \sum_{0 \leq i < t} g_i.$$

Under (1.1), the sequence $(g_{[0,n)})$ diverges to infinity. Moreover, under (1.1), (g_n) is asymptotically equivalent to a monotone sequence, and Karamata's theorem for power series (see Bingham, Goldie and Teugels, 1989, Corollary 1.7.3) asserts that

$$g_{[0,n)} \sim \frac{ng_n}{\gamma} \sim \frac{g(1 - 1/n)}{\Gamma(1 + \gamma)} \quad (1.2)$$

as n tends to infinity. In particular, writing Id for the identity function on the real line, $g(1 - 1/\text{Id})$ is regularly varying of index γ at infinity.

If EX_1 is negative, $ES_n = g_{[0,n)}EX_1$ diverges toward minus infinity as n tends to infinity. It is then conceivable that $\sup_n S_n$ might be almost surely finite, and, if this is the case, our heavy

traffic approximation describes the limiting behavior of this all time supremum when the expectation of the innovations tends to 0. Writing $S_n = (S_n - g_{[0,n)}\mathbb{E}X_1) + \mathbb{E}X_1g_{[0,n)}$, an alternative viewpoint is to consider that

$$F \text{ is centered} \quad (1.3)$$

and seek the limiting behavior of $\sup_{n \geq 0} (X_n - ag_{[0,n)})$ when a tends to 0. As indicated in the first part of this work, this problem has bearing on calculations of ruin probabilities in insurance risk when premiums can barely keep up with claims, in queueing theory, and for moving boundary crossing probabilities as well.

Throughout the paper we will use c for a generic constant whose value may change from place to place.

As in the first part, we assume that

$$F \text{ belongs to the domain of attraction of a stable distribution with index } \alpha \text{ in } (1, 2). \quad (1.4)$$

Assumption (1.4) implies that F is tail balanced in the following sense. Writing F_* for the distribution of $|X_1|$ and $M_{-1}F$ for that of $-X_1$, there exist p and q both in $[0, 1]$, such that $\overline{F} \sim p\overline{F}_*$ and $\overline{M_{-1}F} \sim q\overline{F}_*$ at infinity. These asymptotic relations imply $p+q=1$. Under (1.4), \overline{F}_* is regularly varying of index $-\alpha$; and so is \overline{F} , $\overline{M_{-1}F}$, if p, q , does not vanish respectively. As in the first part of this work, we will assume throughout this paper that p does not vanish. If q vanishes we will also assume that the lower tail of the distribution function F decays slightly faster than the upper one in the sense that for some constant c

$$\overline{M_{-1}F}(t) \leq c\overline{F}(t \log t) \log t \quad \text{ultimately.} \quad (1.5)$$

Assumption (1.4) gives rise to a Lévy measure ν whose density with respect to the Lebesgue measure λ is

$$\frac{d\nu}{d\lambda}(x) = p\alpha x^{-\alpha-1} \mathbb{1}_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1} \mathbb{1}_{(-\infty,0)}(x).$$

The function

$$k(t) = \frac{g_{[0,t)}}{F_*^{\leftarrow}(1-1/t)}$$

will play a role in our following main result — note that the meaning of the notation k in this paper is different than that in part I, but will

play an analogous role. Given (1.2), it is asymptotically equivalent to $g(1 - 1/t)/(\Gamma(1 + \gamma)F_*^{\leftarrow}(1 - 1/t))$ as t tends to infinity.

The first assertion of our main theorem asserts that for the heavy traffic to make sense we should have $\alpha\gamma \geq 1$.

Theorem 1.1. *Assume that (1.1) holds for some positive γ less than 1, and that (1.3) and (1.4) hold. If q vanishes, assume furthermore that (1.5) holds. Then*

(i) *if $\lim_{t \rightarrow \infty} k(t) < \infty$, in particular if $\alpha\gamma < 1$, then for any positive a ,*

$$\sup_{n \geq 0} S_n - ag_{[0,n)} = +\infty$$

in probability.

(ii) *if $\alpha\gamma > 1$ then, writing $\sum_{i \geq 1} \delta_{(t_i, x_i)}$ for a Poisson process with mean intensity $\lambda \otimes \nu$, the distribution of*

$$\frac{\sup_{n \geq 0} (S_n - ag_{[0,n)})}{F_*^{\leftarrow} \left(1 - \frac{1}{k^{\leftarrow}(1/a)} \right)}$$

converges to that of $\sup_{j \geq 0} g_j x_j - t_i^\gamma$. The latter random variable is almost surely finite.

Note that Theorem 1.1 leaves open the boundary case where $\alpha\gamma = 1$ and $\lim_{t \rightarrow \infty} k(t) = +\infty$. It is conceivable that the conclusion of (ii) remains, but we do not know how to prove it. Theorem 1.1 also leaves open the seemingly less interesting situation where k oscillates in such a way that $\liminf_{t \rightarrow \infty} k(t) = 0$ and $\limsup_{t \rightarrow \infty} k(t) = +\infty$.

The following elementary example gives a concrete idea on the rate involved in Theorem 1.1. Using the first part of this work, it is straightforward to extend this example to FARIMA models.

Example. Consider $g_i = i^{\gamma-1}$ with $\gamma < 1$. Assume that $\overline{F}_*(t) \sim ct^{-\alpha}$ as t tends to infinity, and that $\alpha\gamma > 1$. This implies $\overline{F}_*^{\leftarrow}(1 - 1/t) \sim (ct)^{1/\alpha}$ as t tends to infinity. From (1.2) we deduce that $g_{[0,i)} \sim i^\gamma/\gamma$. Thus, $k(t) \sim t^{\gamma-1/\alpha}/(\gamma c^{1/\alpha})$ and $k^{\leftarrow}(1/a) \sim (\gamma c^{1/\alpha}/a)^{\alpha/(\alpha\gamma-1)}$ as a tends to 0. We then obtain $F_*^{\leftarrow}(1 - 1/k^{\leftarrow}(1/a)) \sim (\gamma c^\gamma/a)^{1/(\alpha\gamma-1)}$. Assertion (ii) of Theorem 1.1 implies that $\sup_{n \geq 0} (S_n - ag_{[0,n)})$ grows like $1/a^{1/(\alpha\gamma-1)}$ as a tends to 0.

2. Proof of Theorem 1.1. Let $\Pi = \sum_i \delta_{(t_i, x_i)}$ be a Poisson random measure with mean intensity $\lambda \otimes \nu$. For any positive integer i , set

$$c_{n,i} = g_{[0,i)} \int_{F_*^{\leftarrow}(1/n)}^{F_*^{\leftarrow}(1-1/n)} x \, dF(x).$$

Since F is centered,

$$\int_{F_*^{\leftarrow}(1/n)}^{F_*^{\leftarrow}(1-1/n)} x \, dF(x) = O(F_*^{\leftarrow}(1-1/n)/n)$$

as n tends to infinity. Thus, since γ is less than 1, for any positive M ,

$$\lim_{n \rightarrow \infty} \max_{1 \leq i \leq Mn} c_{n,i}/F_*^{\leftarrow}(1-1/n) = 0. \quad (2.1)$$

Consider the random measure

$$N_n = \sum_{i \geq 1} \delta_{(i/n, S_i/F_*^{\leftarrow}(1-1/n))}$$

in the space of all measures on $[0, \infty) \times (\mathbb{R} \setminus \{0\})$ endowed with the topology of vague convergence. Theorem 5.3 in Barbe and McCormick (2010a) asserts that the distribution of the random measure

$$\sum_{i \geq 1} \delta_{(i/n, (S_i - c_{n,i})/F_*^{\leftarrow}(1-1/n))}$$

converges weakly* to that of $N = \sum_{i \geq 1} \sum_{j \geq 0} \delta_{(t_i, g_j x_i)}$. Since (2.1) holds, this implies that the distribution of N_n converges to that of N as well.

Define $\Lambda = \Lambda(1/a)$ by

$$k(\Lambda) \sim 1/a$$

as a tends to 0, that is $\Lambda \sim k^{\leftarrow}$. Let T be a positive real number. We obtain

$$\begin{aligned} \frac{S_i - ag_{[0,i)}}{F_*^{\leftarrow}(1-1/\Lambda)} &= \frac{S_i - ag_{[0,\Lambda)} \frac{g_{[0,i)}}{g_{[0,\Lambda)}}}{F_*^{\leftarrow}(1-1/\Lambda)} \\ &= \frac{S_i}{F_*^{\leftarrow}(1-1/\Lambda)} - \frac{g_{[0,i)}}{g_{[0,\Lambda)}} (1 + o(1)) \end{aligned}$$

where the $o(1)$ term is uniform in i between 0 and ΛT and as a tends to 0.

Let ϵ be a positive real number. By the Skorokhod-Dudley-Wichura representation theorem, we can construct a version of N and, for each n , a version of N_n such that this version of N_n converges almost surely to N as point measures on $[0, T] \times (\mathbb{R} \setminus \{0\})$. We consider these versions until subsection 2.2, even though we use the same notation as the original processes. For these versions,

$$\sup_{i \geq 0} \frac{S_i \mathbb{1}\{S_i/F_*^{\leftarrow}(1-1/\Lambda) > \epsilon\} - ag_{[0,i]}}{F_*^{\leftarrow}(1-1/\Lambda)} \mathbb{1}\{0 \leq i \leq \Lambda T\} \quad (2.2)$$

converges almost surely to

$$\sup_{i \geq 1} \sup_{j \geq 0} (g_j x_i \mathbb{1}\{g_j x_i > \epsilon\} - t_i^\gamma) \mathbb{1}\{0 \leq t_i \leq T\}.$$

Since ϵ is arbitrary and $\sup_{0 \leq i \leq \Lambda T} (S_i - ag_{[0,i]})/F_*^{\leftarrow}(1-1/\Lambda)$ is within ϵ of (2.2), this implies

$$\lim_{a \rightarrow 0} \sup_{0 \leq i \leq \Lambda T} \frac{S_i - ag_{[0,i]}}{F_*^{\leftarrow}(1-1/\Lambda)} = \sup_{i \geq 1} \sup_{j \geq 0} (g_j x_i - t_i^\gamma) \mathbb{1}\{0 \leq t_i \leq T\}. \quad (2.3)$$

With these preliminaries, we can prove both assertions of Theorem 1.1 in the next two subsections.

2.1. Proof of assertion (i). If x is a real number, we write x_+ for $x \vee 0$. For any nonnegative integer p , define

$$M_p = \max_{i: p \leq t_i < p+1} \sup_{j \geq 0} (g_j x_i)_+.$$

Let T be a positive integer. Since our version of N_n converges almost surely to N ,

$$\begin{aligned} & \max_{0 \leq i < nT} (S_i)_+ / g_{[0,i]} \\ &= \max_{0 \leq p < T} \max_{np \leq i < n(p+1)} \frac{(S_i)_+}{F_*^{\leftarrow}(1-1/n)} \frac{F_*^{\leftarrow}(1-1/n)}{F_*^{\leftarrow}(1-1/i)} \frac{1}{k(i)} \\ &\geq \max_{1 \leq p < T} (M_p + o(1)) \frac{1}{(p+1)^{1/\alpha}} \frac{1+o(1)}{\limsup_{t \rightarrow \infty} |k(t)|} \end{aligned}$$

almost surely. It then suffices to show that

$$\lim_{T \rightarrow \infty} \max_{1 \leq p \leq T} M_p / p^{1/\alpha} = +\infty$$

in probability. Since γ is less than 1, the sequence $(g_j)_{j \geq 0}$ has a largest term, g_∞ , which is positive under (1.1). Let $(\omega_i)_{i \geq 1}$ be a sequence of independent random variables having an exponential distribution with mean 1. The discussion following Lemma 6.1 in Barbe and McCormick (2010a), or the calculation between (2.2.11) and (2.2.12) in this paper, show that $(M_p)_{p \geq 1}$ has the same distribution as $(g_\infty \omega_p^{-1/\alpha})_{p \geq 1}$. Thus, it suffices to show that $\min_{p \geq 1} p \omega_p = 0$ in probability. This follows from the equality

$$\mathbb{P}\left\{\min_{1 \leq p \leq T} p \omega_p > \epsilon\right\} = \prod_{1 \leq p \leq T} e^{-\epsilon/p}$$

and the divergence of the series $\sum_{p \geq 1} 1/p$.

2.2. Proof of assertion (ii). Given our preliminary remarks, and in particular (2.3), it suffices to prove that

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\{\exists n > \Lambda T : S_n > a g_{[0,n)}\} = 0.$$

Arguing as in the beginning of section 3 of Barbe and McCormick (2010b), this is equivalent to showing that

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\{\exists n > \Lambda : S_n > T g_{[0,n)}/k(\Lambda)\} = 0. \quad (2.2.1)$$

This will be achieved in mostly three steps, and a fourth one to handle the part of the proof dealing with the lower tail of the distribution.

Step 1. Let (a_n) and (b_n) be two sequences diverging respectively to $-\infty$ and $+\infty$. We assume that

$$\lim_{n \rightarrow \infty} b_n / (-a_n) \text{ is positive or infinite.}$$

We set

$$\sigma_n^2 = \text{Var}(X_1 \mathbb{1}_{[a_n, b_n]}(X_1))$$

and, for any positive integer i at most n ,

$$Z_{i,n} = \frac{X_i \mathbb{1}_{[a_n, b_n]}(X_i) - \mathbb{E} X_i \mathbb{1}_{[a_n, b_n]}(X_i)}{\sigma_n}.$$

Up to scaling, the part of S_n made by the ‘middle’ innovations is

$$M_n = \sigma_n \sum_{0 \leq i < n} g_i Z_{n-i,n}.$$

As in Barbe and McCormick (2010b), we construct (b_n) to be regularly varying, of the form $F^{\leftarrow}(1 - m_n/n)$ with $1 - m_n/n$ in the range of F and (m_n) being regularly varying of index β .

Proposition 2.2.1. *For any positive β less than 1 and any positive T ,*

$$\lim_{\Lambda \rightarrow \infty} \mathbb{P}\{ \exists n \geq \Lambda : M_n > Tg_{[0,n)}/k(\Lambda) \} = 0.$$

Proof. As in the proof of Proposition 3.1.1 in Barbe and McCormick (2010b),

$$\sigma_n \sim cF^{\leftarrow}(1 - m_n/n)\sqrt{m_n/n}$$

as n tends to infinity. Moreover, for any positive integer r such that $\sum |g_i|^r$ converges,

$$\left| \mathbb{E}\left(\frac{M_n}{\sigma_n \sqrt{n}}\right)^r \right| \leq \frac{c_r}{n}.$$

Markov’s inequality yields

$$\begin{aligned} \mathbb{P}\left\{ M_n > Tg_{[0,n)}/k(\Lambda) \right\} &\leq \left(\frac{\sigma_n \sqrt{n} k(\Lambda)}{Tg_{[0,n)}} \right)^r \frac{c_r}{n} \\ &\sim \frac{c}{T^r} \left(\frac{F^{\leftarrow}(1 - m_n/n)\sqrt{m_n} k(\Lambda)}{g_{[0,n)}} \right)^r \frac{1}{n} \\ &\sim \frac{c}{T^r} k(\Lambda)^r \left(\frac{F^{\leftarrow}(1 - m_n/n)\sqrt{m_n}}{g_{[0,n)}} \right)^r \frac{1}{n}. \end{aligned}$$

This asymptotic equivalent is of the form $k(\Lambda)^r$ times a function of n which is regularly varying of index

$$r\left(\frac{1-\beta}{\alpha} + \frac{\beta}{2} - \gamma\right) - 1 = r\left(\frac{1}{\alpha} - \gamma + \beta\left(\frac{1}{2} - \frac{1}{\alpha}\right)\right) - 1.$$

This index is less than -1 . Therefore, by Bonferroni’s inequality and Karamata’s theorem,

$$\begin{aligned} \mathbb{P}\{ \exists n > \Lambda : M_n > Tg_{[0,n)}/k(\Lambda) \} \\ \leq \frac{c}{T^r} k(\Lambda)^r \left(\frac{F^{\leftarrow}(1 - m_\Lambda/\Lambda)\sqrt{m_\Lambda}}{g_{[0,\Lambda)}} \right)^r. \end{aligned}$$

This bound is regularly varying in Λ of negative index $r\beta((1/2) - (1/\alpha))$ and tends to 0 as Λ tends to infinity. \blacksquare

Step 2. We consider the part of S_n made by the extreme innovations,

$$T_n^+ = \sum_{0 \leq i < n} g_i X_{n-i} \mathbb{1}_{[b_n, \infty)}(X_{n-i}).$$

The purpose of this step is to show that in our problem we can ignore the contribution of T_n^+ in the range of n exceeding $\Lambda^{1+\epsilon}$. The following lemma will be instrumental; it is stronger than what we need in this step, but this strength will turn useful in the next step.

Lemma 2.2.2. *If $\beta < (1 - \gamma)/(1 - 1/\alpha)$ then whenever T is large enough, n is at least Λ and Λ is large enough, $\mathbb{E}T_n^+ \leq Tg_{[0,n]}/k(\Lambda)$.*

Proof. Lemma 3.2.2 in Barbe and McCormick (2010b) implies

$$\mathbb{E}T_n^+ \sim g_{[0,n]} \frac{\alpha}{\alpha - 1} m_n^{1-1/\alpha} \frac{F^{\leftarrow}(1 - 1/n)}{n}.$$

Substituting T by a multiple of it, it suffices to prove that

$$\frac{g_{[0,n]}}{n} m_n^{1-1/\alpha} < T \frac{k(n)}{k(\Lambda)}.$$

Since k is regularly varying of positive index, it suffices to show that

$$\frac{g_{[0,n]}}{n} m_n^{1-1/\alpha} < T.$$

This holds because the left hand side is regularly varying of index

$$\gamma - 1 + \beta(1 - 1/\alpha)$$

which is negative. \blacksquare

The main result of this step 2 is the following.

Proposition 2.2.3. *Let ϵ be a positive real number. If*

$$\beta < \frac{1}{2\gamma} \left(\gamma - \frac{1}{\alpha} \right) \frac{\epsilon}{1 + \epsilon},$$

then

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\{ \exists n > \Lambda^{1+\epsilon} : |T_n^+ - \mathbb{E}T_n^+| > Tg_{[0,n]}/k(\Lambda) \} = 0.$$

Proof. Let $(U_i)_{i \geq 1}$ be a sequence of independent random variables uniform over $[0, 1]$. Let \mathbb{U}_n be the empirical distribution function of $(U_i)_{1 \leq i \leq n}$. We write $(U_{i,n})$ for the order statistics of $(U_i)_{1 \leq i \leq n}$. Without any loss of generality, we assume that $X_i = F^{\leftarrow}(1 - U_i)$, so that, for any n large enough

$$\begin{aligned} T_n^+ &= \sum_{0 \leq i < n} g_i F^{\leftarrow}(1 - U_{n-i}) \mathbb{1}\{U_{n-i} \leq m_n/n\} \\ &\leq c F^{\leftarrow}(1 - U_{1,n}) g_{[0,n\mathbb{U}_n(m_n/n)]}. \end{aligned}$$

Let $(\xi_n)_{n \geq 1}$ be a slowly varying nondecreasing sequence such that $\sum_{n \geq 1} 1/n\xi_n$ converges. From Kiefer's (1972) Theorem 1 we deduce that $U_{1,n} \geq 1/\xi_n$ almost surely for n large enough, while Shorack and Wellner's (1978) Theorem 2 implies $\mathbb{U}_n \leq \xi_n \text{Id}$. Then, using Potter's bound for $F^{\leftarrow}(1 - 1/\text{Id})$ and using that $(g_{[0,n]})$ is regularly varying, we obtain

$$T_n^+ \leq c F^{\leftarrow}(1 - 1/n) \xi_n^{2/\alpha} g_{[0,m_n]}.$$

Therefore, the inequality $T_n^+ > Tg_{[0,n]}/k(\Lambda)$ implies, almost surely for n large enough,

$$k(\Lambda) > cT \frac{k(n)}{g_{[0,m_n]} \xi_n^{2/\alpha}}. \quad (2.2.2)$$

In this inequality, the right hand side is regularly varying of index $\gamma - (1/\alpha) - \beta\gamma$, which is positive provided $\beta < (\gamma - (1/\alpha))/\gamma$. In this range of β and in the range of n at least $\Lambda^{1+\epsilon}$, the right hand side of (2.2.2) is at least a constant times its value at $\Lambda^{1+\epsilon}$. This lower bound, as a function of Λ , is regularly varying and for (2.2.2) to hold we must have, comparing the index of regular variation,

$$\gamma - \frac{1}{\alpha} \geq (1 + \epsilon) \left(\gamma - \frac{1}{\alpha} - \beta\gamma \right).$$

This does not hold under the assumption of the lemma, and therefore (2.2.2) does not occur. So $T_n^+ \leq Tg_{[0,n]}/k(\Lambda)$ almost surely in the range $n \geq \Lambda^{1+\epsilon}$ and for Λ large enough.

Lemma 2.2.2 implies that

$$ET_n^+ \leq Tg_{[0,n)}/k(\Lambda),$$

whenever n exceeds $\Lambda^{1+\epsilon}$ and Λ is large enough. This proves the proposition. \blacksquare

Step 3. Given Lemma 2.2.2, our goal is now to show that

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P} \left\{ \exists n \in (\Lambda, \Lambda^{1+\epsilon}) : T_n^+ > Tg_{[0,n)}/k(\Lambda) \right\} = 0. \quad (2.2.3)$$

For this, we approximate T_n^+ by a simpler quantity.

It is convenient to write N for $\Lambda^{1+\epsilon}$. Let $(U_i)_{i \geq 1}$ be a sequence of independent random variables uniformly distributed on $[0, 1]$. Let τ be the random permutation of $\{1, 2, \dots, N\}$ such that $U_{\tau(i)} = U_i$. Without any loss of generality we can assume that $X_i = F^{\leftarrow}(1 - U_i)$. For n in $(\Lambda, \Lambda^{1+\epsilon})$, we then have

$$\begin{aligned} T_n^+ &= \sum_{1 \leq i \leq n} g_{n-i} X_i \mathbb{1}\{X_i > b_n\} \\ &= \sum_{1 \leq i \leq N} g_{n-\tau(i)} F^{\leftarrow}(1 - U_{i,N}) \mathbb{1}\{U_{i,N} \leq m_n/n\} \mathbb{1}\{\tau(i) \leq n\}. \end{aligned}$$

Let $(V_i)_{i \geq 1}$ be a sequence of independent random variables uniformly distributed on $[0, 1]$, independent of (U_i) . Let G_N be the empirical distribution function of $(V_i)_{1 \leq i \leq N}$. Without any loss of generality we can assume that $\tau(i) = NG_N(V_i)$. Then, setting

$$\mathcal{R}_{1,n,N} = \left\{ i : U_{i,N} \leq \frac{m_n}{n}; G_N(V_i) \leq \frac{n}{N} \right\},$$

we have

$$T_n^+ = \sum_{i \in \mathcal{R}_{1,n,N}} g_{n-NG_N(V_i)} F^{\leftarrow}(1 - U_{i,N}).$$

Let $i^* = i_{n,N}^*$ be in $\mathcal{R}_{1,n,N}$ such that $n - NG_N(V_i)$ is minimum when i is i^* . Such i^* exists and is well defined because $\mathcal{R}_{1,n,N}$ is finite and for i in $\mathcal{R}_{1,n,N}$ the differences $n - NG_N(V_i)$ are nonnegative and assume, almost surely, different values for different i . Set

$$T_{1,n,N}^+ = g_{n-NG_N(V_{i^*})} F^{\leftarrow}(1 - U_{i^*,N}).$$

Our next lemma shows that we can approximate T_n^+ by $T_{1,n,N}^+$ in order to prove (2.2.3).

Lemma 2.2.4. *For any ϵ and β small enough, for any positive T ,*

$$\lim_{\Lambda \rightarrow \infty} \mathbb{P}\{\exists n \in (\Lambda, \Lambda^{1+\epsilon}) : |T_n^+ - T_{1,n,N}^+| > Tg_{[0,n)}/k(\Lambda)\} = 0.$$

Proof. Writing $\#\mathcal{R}_{1,n,N}$ for the cardinality of $\mathcal{R}_{1,n,N}$, we have

$$\begin{aligned} & |T_n^+ - T_{1,n,N}^+| \\ &= \sum_{i \in \mathcal{R}_{1,n,N} \setminus \{i^*\}} g_{n-NG_N(V_i)} F^{\leftarrow}(1 - U_{i,N}) \\ &\leq \#\mathcal{R}_{1,n,N} \max_{i \in \mathcal{R}_{1,n,N} \setminus \{i^*\}} g_{n-NG_N(V_i)} \max_{i \in \mathcal{R}_{1,n,N}} F^{\leftarrow}(1 - U_{i,N}). \end{aligned} \quad (2.2.4)$$

Let η be a positive real number less than 1. As in Barbe and McCormick (2010b), let (W_i) be a random walk whose increments are standard exponential random variables, and write $U_{i,N}$ as W_i/W_{N+1} . Lemma 3.4.3 in Barbe and McCormick (2010b) shows that

$$\max_{\Lambda \leq n \leq N} \#\mathcal{R}_{1,n,N} = O_P(m_N \log N)$$

as Λ tends to infinity.

Robbins (1954) implies that provided c is small enough, the set

$$\Omega = \{U_{i,N} \geq ci/N : 1 \leq i \leq N\}$$

has probability at least $1 - \eta$. An integer i in $\mathcal{R}_{1,n,N}$ is such that $U_{i,N} \leq m_n/n$, and on Ω we obtain $i \leq cm_n N/n$. So, if i is in $\mathcal{R}_{1,n,N} \setminus \{i^*\}$ and Ω occurs,

$$|V_{i^*} - V_i| \geq \min_{2 \leq j \leq cm_n N/n} V_{j, \lfloor cm_n N/n \rfloor} - V_{j-1, \lfloor cm_n N/n \rfloor}. \quad (2.2.5)$$

Theorem 3.1 in Devroye (1982) implies that the right hand side of (2.2.5) is almost surely at least $n^2/cm_n^2 N^2 \log N$ whenever Λ is large enough. For n at least Λ , using that n/m_n is asymptotically equivalent to a nondecreasing function and hence at least Λ/m_Λ , the right hand side of (2.2.5) is at least $c\Lambda^{-2(\beta+\epsilon)}/\log \Lambda$, and, if

β and ϵ are small enough, dominates $1/\sqrt{N}$ asymptotically. Since $G_N = \text{Id} + O_P(1/\sqrt{N})$ by Donsker's (1952) invariance principle,

$$\min_{i \in \mathcal{R}_{1,n,N} \setminus \{i^*\}} N |G_N(V_i) - G_N(V_{i^*})| \gtrsim \frac{cn^2}{m_n^2 N \log N}$$

for all n in (Λ, N) , with probability at least $1 - \eta$. Thus, writing $n - NG_N(V_i)$ as $n - NG_N(V_{i^*}) + N(G_N(V_{i^*}) - G_N(V_i))$, it follows that

$$\max_{i \in \mathcal{R}_{1,n,N} \setminus \{i^*\}} g_{n-NG_N(V_i)} \lesssim cg_{n^2/(m_n^2 N \log N)} \leq cg_{\Lambda^2/m_\Lambda^2 N \log N}.$$

Since $P\{U_{1,N} \geq c/N\} \geq 1 - \eta$ if c is small enough and N is large enough,

$$F^{\leftarrow}(1 - U_{1,N}) \leq cF^{\leftarrow}(1 - 1/N)$$

with probability at least $1 - \eta$. Using Potter's bound, this is at most $cF^{\leftarrow}(1 - 1/n)(N/n)^{(1/\alpha)+\eta}$.

Thus, with probability at least $1 - \eta$, (2.2.4) is at most

$$cm_N(\log N)g_{\frac{\Lambda^2}{m_n^2 N \log N}} \left(\frac{N}{n}\right)^{(1/\alpha)+\eta} F^{\leftarrow}(1 - 1/n).$$

For this bound to exceed $Tg_{[0,n]}/k(\Lambda)$ we must have, as Λ tends to infinity,

$$cm_N(\log N)g_{\frac{\Lambda^2}{m_\Lambda^2 N \log N}} \left(\frac{N}{\Lambda}\right)^{(1/\alpha)+\eta} \geq T \frac{k(n)}{k(\Lambda)} \gtrsim T. \quad (2.2.6)$$

The left hand side is a regularly varying of Λ , of index

$$\beta(1 + \epsilon) + (1 - 2\beta - \epsilon)(\gamma - 1) + \epsilon \left(\frac{1}{\alpha} + \eta\right) = \gamma - 1 + O(\beta) + O(\epsilon).$$

This index is negative if β and ϵ are small enough, and (2.2.6) cannot hold. This proves the lemma. \blacksquare

Note that by construction $T_{1,n,N}^+$ is an approximation of the sum $\sum_{0 \leq i < n} g_i X_{n-i} \mathbb{1}\{X_{n-i} > b_n\}$ by a single one of its summands. Since each summand is at most $|g|_\infty X_{n,n}$, we see that in order to show that

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} P\{\exists n \in (\Lambda, \Lambda^{1+\epsilon}) : T_{1,n,N}^+ > Tg_{[0,n]}/k(\Lambda)\} = 0,$$

it suffices to prove that

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\{ \exists n \in (\Lambda, \Lambda^{1+\epsilon}) : X_{n,n} > Tg_{[0,n]}/k(\Lambda) \} = 0.$$

Writing $X_{n,n} = F^{\leftarrow}(1 - U_{1,n})$, this amounts to proving that

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\left\{ \exists n \in (\Lambda, \Lambda^{1+\epsilon}) : U_{1,n} \leq \overline{F}\left(T \frac{k(n)}{k(\Lambda)} F^{\leftarrow}(1 - 1/n)\right) \right\} = 0. \quad (2.2.7)$$

Let (V_i) be a new sequence of independent random variables having a uniform distribution over $[0, 1]$. Write $(V_{i,n})_{1 \leq i \leq n}$ for the order statistics of $(V_i)_{i \leq n}$. Setting $V_{1,0} = 1$, we have $(U_{1,n})_{n \geq \Lambda} \stackrel{d}{=} (U_{1,\Lambda} \wedge V_{1,n-\Lambda})_{n \geq \Lambda}$. Applying Bonferroni's inequality, we see that for (2.2.7) to hold it suffices to have

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\left\{ \exists n \in (\Lambda, \Lambda^{1+\epsilon}) : U_{1,\Lambda} \leq \overline{F}\left(T \frac{k(n)}{k(\Lambda)} F^{\leftarrow}(1 - 1/n)\right) \right\} = 0 \quad (2.2.8)$$

and, replacing n by $\Lambda + n$, and setting

$$v_n = \overline{F}\left(T \frac{k(\Lambda + n)}{k(\Lambda)} F^{\leftarrow}\left(1 - \frac{1}{\Lambda + n}\right)\right)$$

to also have

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\left\{ \exists n \in (0, \Lambda^{1+\epsilon}) : V_{1,n} \leq v_n \right\} = 0. \quad (2.2.9)$$

The right hand side of the inequality involved in (2.2.8) is equivalent to a function decreasing in n . So in the range of n between Λ and $\Lambda^{1+\epsilon}$ it is at most a constant times

$$T^{-\alpha} \overline{F}\left(F^{\leftarrow}\left(1 - \frac{1}{\Lambda}\right)\right) \sim \frac{T^{-\alpha}}{\Lambda}.$$

Since the distribution of $\Lambda U_{1,\Lambda}$ converges to a standard exponential one, (2.2.8) holds.

To prove that (2.2.9) holds, we use a blocking argument. Consider a real number θ greater than $1/(\alpha\gamma - 1)$ and for any integer p set

$n_p = \lfloor \Lambda p^\theta \rfloor$. Potter's bound implies as Λ tends to infinity and uniformly in p positive

$$\begin{aligned} v_{n_p} &\lesssim T^{-\alpha} \overline{F} \left((1 + p^\theta)^{\gamma - (1/\alpha) - \eta} F^{\leftarrow} \left(1 - \frac{1}{\Lambda(1 + p^\theta)} \right) \right) \\ &\lesssim T^{-\alpha} \overline{F} \left((1 + p^\theta)^{\gamma - 2\eta} F^{\leftarrow} \left(1 - \frac{1}{\Lambda} \right) \right) \\ &\lesssim T^{-\alpha} (1 + p^\theta)^{-\alpha(\gamma - 3\eta)} \frac{1}{\Lambda}. \end{aligned}$$

Therefore, for any Λ large enough and any positive p ,

$$\begin{aligned} \mathbb{P}\{V_{1,n_p} \leq v_{n_p}\} &\leq \mathbb{P}\left\{V_{1,n_p} \leq cT^{-\alpha}(1+p)^{-\theta\alpha(\gamma-3\eta)} \frac{1}{\Lambda}\right\} \\ &\leq 1 - \left(1 - cT^{-\alpha}(1+p)^{-\theta\alpha(\gamma-3\eta)} \frac{1}{\Lambda}\right)^{\lfloor \Lambda p^\theta \rfloor}. \end{aligned} \quad (2.2.10)$$

Note that $(1+p)^{-\theta\alpha(\gamma-3\eta)}/\Lambda$ tends to 0 as Λ tends to infinity, uniformly in p nonnegative. So (2.2.10) is at most

$$c\Lambda p^\theta T^{-\alpha}(1+p)^{-\theta\alpha(\gamma-3\eta)} \frac{1}{\Lambda} \lesssim cT^{-\alpha} p^{\theta(1-\alpha\gamma+3\alpha\eta)}.$$

Given our choice of θ , we see that if η is small enough, the exponent of p is less than -1 . Thus, Bonferroni's inequality implies

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\{\exists p \geq 1 : V_{1,n_p} \leq v_{n_p}\} = 0.$$

If n is between n_{p-1} and n_p , then $V_{1,n} \geq V_{1,n_p}$ and since $v_{n_p}/v_{n_{p-1}}$ is bounded away from 0 and infinity uniformly in p as Λ tends to infinity, we proved (2.2.9) and (2.2.7) as well. \blacksquare

Step 4. Let $T_n^- = \sum_{0 \leq i < n} g_i X_{n-i} \mathbb{1}_{(-\infty, a_n)}(X_{n-i})$. Combining steps 1, 2 and 3, we see that

$$\begin{aligned} \lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\{\exists n > \Lambda : S_n - T_n^- - \mathbb{E}(S_n - T_n^-) > Tg_{[0,n]}/k(\Lambda)\} \\ = 0. \end{aligned}$$

Hence, in order to prove (2.2.1), it suffices to show that

$$\lim_{T \rightarrow \infty} \limsup_{\Lambda \rightarrow \infty} \mathbb{P}\{\exists n > \Lambda : |T_n^- - \mathbb{E}T_n^-| > Tg_{[0,n]}/k(\Lambda)\} = 0.$$

This follows by the very same arguments as in section 3.5 of Barbe and McCormick (2010b).

It remains to show that the limiting random variable involved in Theorem 1.1.ii is almost surely finite. We write ν_+ and ν_- for the restriction of ν to $(-\infty, 0)$ and $(0, \infty)$ respectively. Let N_+ and N_- be two independent Poisson processes with respective mean measures $\lambda \otimes \nu_-$ and $\lambda \otimes \nu_+$. For a point process $N = \sum_{i \geq 1} \delta_{(t_i, x_i)}$ write N^g for $\sup_{j \geq 0} g_j x_i - t_i^\gamma$. Since N_- and N_+ are independent, $N_- + N_+$ is a Poisson process with mean intensity $\lambda \otimes \nu$. Since $(N_- + N_+)^g = N_+^g \vee N_-^g$, it suffices to show that N_+^g is finite. Write $N_+ = \sum_{i \geq 1} \delta_{(t_i, x_i)}$. Since (g_j) is bounded,

$$N_+^g \leq \sup_{i \geq 1} c x_i - t_i^\gamma \quad (2.2.11)$$

whenever c is at least $\max_{j \geq 0} g_j$. So it suffices to show that the upper bound in (2.2.11) is almost surely finite.

Since N_+ is a Poisson process, the random variables

$$M_k = \sup_{i: t_i \in [k, k+1)} c x_i - k^\gamma, \quad k \geq 0,$$

are independent. Moreover,

$$\sup_{i \geq 1} c x_i - t_i^\gamma \leq \sup_{k \geq 0} M_k.$$

Recall that N_+ has intensity $\lambda \otimes \nu_+$. Since

$$\begin{aligned} \mathbb{P}\{M_k > x\} &= \mathbb{P}\left\{\exists i : (t_i, x_i) \in [k, k+1) \times \left(\frac{x + k^\gamma}{c}, \infty\right)\right\} \\ &= \mathbb{P}\left\{N\left([k, k+1) \times \left(\frac{x + k^\gamma}{c}, \infty\right)\right) \geq 1\right\} \\ &= 1 - \exp\left(-p\left(\frac{x + k^\gamma}{c}\right)^{-\alpha}\right), \end{aligned}$$

we have

$$\mathbb{P}\left\{\max_{k \geq 1} M_k > x\right\} \leq \sum_{k \geq 0} \left(1 - \exp\left(-p\left(\frac{c}{x + k^\gamma}\right)^\alpha\right)\right) \quad (2.2.12)$$

This series is convergent since its k -th term is equivalent to $c/k^{\alpha\gamma}$ as k tends to infinity and $\alpha\gamma$ is greater than 1. Bounding $c/(x + k^\gamma)$

by $c/(1+k^\gamma)$ when x exceeds 1, the dominated convergence theorem implies that (2.2.12) tends to 0 as x tends to infinity, concluding the proof of Theorem 1.1.

References.

- Ph. Barbe, W.P. McCormick (2010a). Invariance principles for some FARIMA and nonstationary linear processes in the domain of attraction of a stable distribution, [arXiv:1007.0576](#).
- Ph. Barbe, W.P. McCormick (2010b). Ruin probabilities in tough times – Part 1: heavy-traffic approximation for fractionally integrated random walks in the domain of attraction of a nonGaussian stable distribution, [arXiv:1101.4437](#).
- N.H. Bingham, C.M. Goldie, J.L. Teugels (1989). *Regular Variation*, 2nd ed. Cambridge University Press.
- M. Donsker (1952). Justification and extension of Doob’s heuristic approach to the Kolmogorov-Smirnov theorems, *Ann. Math. Statist.*, 23, 277–281.
- J. Kiefer (1972). Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$, *Proc. Sixth Berkeley Sympos. on Math. Statist. and Probab.*, 1, 227–244.
- H. Robbins (1954). A one-sided confidence interval for an unknown distribution function, *Ann. Math. Statist.*, 25, 409.
- G.R. Shorack, J.A. Wellner (1978). Linear bounds on the empirical distribution function, *Ann. Probab.*, 6, 349–353.

Ph. Barbe
90 rue de Vaugirard
75006 PARIS
FRANCE
philippe.barbe@math.cnrs.fr

W.P. McCormick
Dept. of Statistics
University of Georgia
Athens, GA 30602
USA
bill@stat.uga.edu